FINITE DIMENSIONAL QUASI-HOPF ALGEBRAS WITH RADICAL OF CODIMENSION 2

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1. Introduction

It is shown in [EO], Proposition 2.17, that a finite dimensional quasi-Hopf algebra with radical of codimension 1 is semisimple and 1-dimensional. On the other hand, there exist quasi-Hopf (in fact, Hopf) algebras, whose radical has codimension 2. Namely, it is known [N] that these are exactly the Nichols Hopf algebras H_{2^n} of dimension 2^n, $n \geq 1$ (one for each value of n).

The main result of this paper is that if H is a finite dimensional **quasi-Hopf algebra** over $\mathbb{C}$ with radical of codimension 2, then H is twist equivalent to a Nichols Hopf algebra H_{2^n}, $n \geq 1$, or to a lifting of one of the four special quasi-Hopf algebras $H(2)$, $H_+(8)$, $H_-(8)$, $H(32)$ of dimensions 2, 8, 8, and 32, defined in Section 3. As a corollary we obtain that any finite tensor category which has two invertible objects and no other simple object is equivalent to $\mathrm{Rep}(H_{2^n})$ for a unique $n \geq 1$, or to a deformation of the representation category of $H(2)$, $H_+(8)$, $H_-(8)$, or $H(32)$. In the case of $H(2)$ such a lifting (deformation) must clearly be trivial; for the other three cases we plan to study possible liftings in a later publication.

As another corollary we prove that any nonsemisimple quasi-Hopf algebra of dimension 4 is twist equivalent to H_4.

Thus, this paper should be regarded as a beginning of the structure theory of finite dimensional basic quasi-Hopf algebras. It is our expectation that this theory will be a nontrivial generalization of the deep and beautiful theory of finite dimensional basic (or, equivalently, pointed) Hopf algebras (see [AS] and references therein).

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2. Preliminaries

All constructions in this paper are done over the field of complex numbers $\mathbb{C}$.

2.1. Recall that a finite rigid tensor category $\mathcal{C}$ over $\mathbb{C}$ is a rigid tensor category over $\mathbb{C}$ with finitely many simple objects and enough projectives such that the neutral object $\mathbf{1}$ is simple (see [EO]). Let $\mathrm{Irr}(\mathcal{C})$ denote the (finite) set of isomorphism classes of simple objects of $\mathcal{C}$. Then to each object $X \in \mathcal{C}$ there is attached a

positive number $\mathrm{FPdim}(X)$, called the Frobenius-Perron (FP) dimension of X, and the Frobenius-Perron (FP) dimension of $\mathcal{C}$ is defined by

$$\mathrm{FPdim}(\mathcal{C}) = \sum_{X \in \mathrm{Irr}(\mathcal{C})} \mathrm{FPdim}(X)\mathrm{FPdim}(P_X),$$

where P_X denotes the projective cover of X [E]. By Proposition 2.6 in [EO], $\mathcal{C}$ is equivalent to $\mathrm{Rep}(H)$, H a finite dimensional quasi-Hopf algebra, if and only if the FP-dimensions of its objects are integers. We refer the reader to [D] for the definition of a quasi-Hopf algebra and the notion of twist (= gauge) equivalence between quasi-Hopf algebras.

2.2. Recall that Nichols' Hopf algebra H_{2^n} $(n \geq 1)$ is the 2^n-dimensional Hopf algebra generated by $g, x_1, \ldots, x_{n-1}$ with relations $g^2 = 1$, $x_i^2 = 0$ $(1 \leq i \leq n-1)$, $x_i x_j = -x_j x_i$ $(1 \leq i \neq j \leq n-1)$ and $g x_i = -x_i g$, where g is a grouplike element and $\Delta(x_i) = x_i \otimes 1 + g \otimes x_i$ [N]. The Hopf algebra H_4 is known as the Sweedler's Hopf algebra [S]. Nichols' Hopf algebras are the unique (up to isomorphism) Hopf algebras with radical of codimension 2. It is well known (and easy to check) that H_{2^n} is self dual.

2.3. The following is the simplest (and well known) example of a quasi-Hopf algebra not twist equivalent to a Hopf algebra. The 2-dimensional quasi-Hopf algebra $H(2)$ is generated by a grouplike element g such that $g^2 = 1$, with associator $\Phi = 1 - 2p_- \otimes p_- \otimes p_-$ (where $p_+ := (1+g)/2, p_- := (1-g)/2$), distinguished elements $\alpha = g$, $\beta = 1$, and antipode $S(g) = g$. It is not twist equivalent to a Hopf algebra, and any 2-dimensional quasi-Hopf algebra is well known to be twist equivalent either to $\mathbb{C}[\mathbb{Z}_2]$ or to $H(2)$.

2.4. Let H be a finite dimensional algebra, and let $I := \mathrm{Rad}(H)$ be the Jacobson radical of H. Then we have a filtration on H by powers of I. So one can consider the associated graded algebra $\mathrm{gr}(H) = \bigoplus_{k \geq 0} H_k$, $H_k := I^k/I^{k+1}$ $(I_0 := H)$.

The following lemma is standard.

Lemma 2.1. *(i) Let L be any linear complement of I^2 in H. Then L generates H as an algebra.*

(ii) $gr(H)$ is generated by H_0 and H_1.

Proof. (i) Let $B \subseteq H$ be the subalgebra generated by L. We will show by induction in k that $B + I^k = H$ for all k. This implies the statement, since $I^N = 0$ for some N. The base of induction, $k = 2$, is clear. So assume that $k > 2$. By the induction assumption, all we need to show is that $I^{k-1} \subseteq B + I^k$. So let us take $a \in I^{k-1}$, and show that $a \in B + I^k$. We may assume that $a = i_1 \cdots i_{k-1}$, where $i_m \in I$ for all m. Let $\bar{i}_m \in L \cap I$ be the unique elements such that $\bar{i}_m - i_m \in I^2$. Then $\bar{a} := \bar{i}_1 \cdots \bar{i}_{k-1} \in B$, and $a - \bar{a} \in I^k$. We are done.

(ii) Clearly, $\mathrm{gr}(I)$ is the radical of $\mathrm{gr}(H)$. Thus we only need to show that $\mathrm{gr}(I^2) = \mathrm{gr}(I)^2$ (then everything follows from (i)). The inclusion $\mathrm{gr}(I^2) \supseteq \mathrm{gr}(I)^2$ is clear, so let us prove the opposite inclusion. Since both spaces are graded, it suffices to pick a homogeneous element $a \in \mathrm{gr}(I^2)$ and show that it lies in $\mathrm{gr}(I)^2$. Suppose a has degree k, i.e. $a \in I^k/I^{k+1}$. Let $\tilde{a}$ be a lifting of a to I^k. Then there exist elements $i_{1j}, ..., i_{kj} \in I$, $j = 1, ..., m$, such that $\tilde{a} = \sum_j i_{1j} \cdots i_{kj}$. Let $\bar{i}_{1j}, ..., \bar{i}_{kj}$ be the projections of $i_{1j}, ..., i_{kj}$ to $I/I^2 \subseteq \mathrm{gr}(I)$. Then $a = \sum_j \bar{i}_{1j} \cdots \bar{i}_{kj}$. Since $k \geq 2$, we are done. $\square$

2.5. Let H be a basic quasi-Hopf algebra (i.e., all irreducible H-modules are 1-dimensional) and let $I := \mathrm{Rad}(H)$ be the radical of H. Then I is a quasi-Hopf ideal (i.e. $\Delta(I) \subseteq H\otimes I + I\otimes H$, $S(I) = I$ and $\varepsilon(I) = 0$), so we have a quasi-Hopf algebra filtration on H by powers of I. The associated graded algebra $\mathrm{gr}(H) = \bigoplus_k H_k$ is thus also a quasi-Hopf algebra. We have $H_0 = \mathrm{Fun}(G)$, where G is a finite group (of characters of H), $H_1 = \bigoplus_{a,b\in G} \mathrm{Ext}^1(a,b)^*$. Note that by definition, the associator of $\mathrm{gr}(H)$ lives in $H_0^{\otimes 3}$, so it corresponds to an element of $H^3(G,\mathbb{C}^*)$.

3. The main results

Our first main theorem is the following.

Theorem 3.1. *Let H be a finite dimensional quasi-Hopf algebra over $\mathbb{C}$ with radical of codimension 2. Suppose that the associator of H is trivial on $1-$dimensional H-modules (i.e., the subcategory of Rep(H) generated by $1-$dimensional objects is equivalent to Rep$(\mathbb{Z}_2)$ with the trivial associator). Then H is twist equivalent to a Nichols Hopf algebra H_{2^n}, $n \geq 1$.*

The proof of the theorem is given in Section 4.

The assumption on the triviality of the associator of H on 1-dimensional modules is clearly essential already for $\dim(H) = 2$, because of the existence of the quasi-Hopf algebra $H(2)$. However, this assumption is also essential in higher dimensions, as the following propositions show.

Proposition 3.2. *There exist two $8-$dimensional quasi-Hopf algebras $H_\pm(8)$ (permuted by the action of the Galois group), which have the following structure. As algebras $H_\pm(8)$ are generated by g, x with the relations $gx = -xg$, $g^2 = 1$, $x^4 = 0$. The element g is grouplike, while the coproduct of x is given by the formula*

$$\Delta(x) = x\otimes(p_+ \pm ip_-) + 1\otimes p_+x + g\otimes p_-x,$$

where $p_+ := (1+g)/2, p_- := (1-g)/2$. The associator is $\Phi = 1 - 2p_-\otimes p_-\otimes p_-$, the distinguished elements are $\alpha = g$, $\beta = 1$, and the antipode is $S(g) = g$, $S(x) = -x(p_+ \pm ip_-)$.

Proposition 3.3. *There exists a $32-$ dimensional quasi-Hopf algebra $H(32)$, which has the following structure. As an algebra $H(32)$ is generated by g, x, y with the relations $gx = -xg$, $gy = -yg$, $g^2 = 1$, $x^4 = 0$, $y^4 = 0$, $xy + iyx = 0$. The element g is grouplike, while the coproducts of x, y are given by the formulas*

$$\Delta(x) = x\otimes(p_+ + ip_-) + 1\otimes p_+x + g\otimes p_-x,$$

$$\Delta(y) = y\otimes(p_+ - ip_-) + 1\otimes p_+y + g\otimes p_-y,$$

where $p_+ := (1+g)/2, p_- := (1-g)/2$. The associator is $\Phi = 1 - 2p_-\otimes p_-\otimes p_-$, the distinguished elements are $\alpha = g$, $\beta = 1$, and the antipode is $S(g) = g$, $S(x) = -x(p_+ + ip_-)$, $S(y) = y(p_+ - ip_-)$. Thus, $H(32)$ is generated by its quasi-Hopf subalgebras $H_+(8)$ and $H_-(8)$ generated by g, x and g, y, respectively.

Propositions 3.2 and 3.3 are proved in Section 5.

Nonetheless, it turns out that outside of dimensions 2, 8 and 32 the assumption of the triviality of the associator of H on 1-dimensional modules is satisfied automatically. This striking fact follows from our second main theorem, which is the following.

Theorem 3.4. *Let H be a finite dimensional quasi-Hopf algebra over $\mathbb{C}$ with radical of codimension 2. Suppose that the associator of H is nontrivial on $1-$dimensional H-modules (i.e., the subcategory of Rep(H) generated by $1-$dimensional objects is equivalent to Rep$(H(2))$). Then gr(H) is twist equivalent to $H(2)$, $H_+(8)$, $H_-(8)$, or $H(32)$; in particular, the dimension of H is 2, 8 or 32. The quasi-Hopf algebras $H_+(8)$ and $H_-(8)$ are not twist equivalent.*

Theorem 3.4 is proved in Section 6.

Corollary 3.5. *Any finite tensor category which has two invertible objects and no other simple object is tensor equivalent to Rep(H_{2^n}) for a unique $n \geq 1$, or to a deformation of the representation category of $H(2)$, $H_+(8)$, $H_-(8)$, or $H(32)$.*

Proof. The FP-dimension of any object in $\mathcal{C}$ is an integer. Therefore, by Proposition 2.6 in [EO], $\mathcal{C}$ is equivalent to a representation category of a finite dimensional quasi-Hopf algebra with radical of codimension 2. The result follows now from Theorems 3.1 and 3.4. $\square$

Corollary 3.6. *Any nonsemisimple quasi-Hopf algebra of dimension 4 is twist equivalent to H_4.*

Proof. By dimension counting it follows that H is basic; that is, all irreducible representations of H are $1-$dimensional. Moreover, by Theorem 2.17 in [EO], H has more than one irreducible representation. Let χ be a non-trivial $1-$dimensional representation of H, and let P_χ denote the projective cover of χ. Then $P_\chi = \chi \otimes P_\varepsilon$, where P_ε is the projective cover of the trivial representation ε. Therefore 4 equals $\dim(P_\varepsilon)$ times the number of simple objects in Rep(H). Since both numbers are greater than 1, we have that they are equal to 2. So H has 2 irreducible representations ε and χ, with $\chi^2 = \varepsilon$. The result follows now from Theorems 3.1 and 3.4. $\square$

Corollary 3.7. *A nonsemisimple finite tensor category $\mathcal{C}$ of FP-dimension 4 is tensor equivalent to Rep(H_4).*

Proof. It suffices to show that $\mathcal{C}$ has integer FP-dimensions of objects; in this case by Proposition 2.6 in [EO], $\mathcal{C}$ is a representation category of a 4-dimensional quasi-Hopf algebra, and Corollary 3.6 applies.

Let X be a simple object of $\mathcal{C}$ of FP-dimension d. If $X \otimes X^* \neq \mathbf{1}$ then $X \otimes X^*$ contains as constituents $\mathbf{1}$ and another object. So if $d \neq 1$ then $d \geq \sqrt{2}$. On the other hand, the projective cover $P_{\mathbf{1}}$ of the neutral object has to involve other objects, so it is at least 2-dimensional. This shows that the only chance for $\mathcal{C}$ to have dimension 4 is when all simple objects are 1-dimensional. We are done. $\square$

Remark 3.8. We note that *semisimple* finite tensor categories of FP-dimension 4 are easy to classify. Such a category is either the category of modules over the group algebra of a group of order 4 with associativity defined by a 3-cocycle on this group, or a Tambara-Yamagami category [TY] corresponding to the Ising model (see [ENO], Proposition 8.32).

4. Proof of Theorem 3.1

By assumption, the associator of gr(H) is trivial. Therefore gr(H) is twist equivalent to a Hopf algebra, and by the result of Nichols [N], gr(H) is twist equivalent to a Nichols Hopf algebra $A := H_{2^n}$.

From this point we will identify $\mathrm{gr}(H)$ and A as quasi-Hopf algebras.

We will now show that H itself is twist equivalent to a Nichols Hopf algebra, completing the proof of the theorem.

Let I_r be the radical of $H^{\otimes r}$. Then of course $I_r = \sum_{k=1}^{r} H^{\otimes k-1} \otimes I \otimes H^{\otimes r-k}$.

Let Φ be the associator of H. Then $\Phi = 1 + \phi$, where $\phi \in I_3$. Assume that $\phi \in I_3^m$, but $\phi \notin I_3^{m+1}$. We will show that by twisting we can change ϕ so that it will belong to I_3^{m+1}. Then by a chain of twists we can make sure that $\phi = 0$, and by Nichols' result we are done.

Let ϕ' be the projection of ϕ to $I_3^m/I_3^{m+1} = A^{\otimes 3}[m]$. Obviously, ϕ' is a $3-$cocycle of A^* with trivial coefficients. But $A^* = A = \mathbb{C}[\mathbb{Z}_2] \ltimes \Lambda V$, and

$$H^3(A, \mathbb{C}) = H^3(\mathbb{C}[\mathbb{Z}_2] \ltimes \Lambda V, \mathbb{C}) = H^3(\Lambda V, \mathbb{C})^{\mathbb{Z}_2} = (S^3 V^*)^{\mathbb{Z}_2} = 0,$$

since $\mathbb{Z}_2$ acts on V by sign. Thus there exists an element $j' \in A^{\otimes 2}[m]$ such that $dj' = \phi'$. Let j be a lifting of j' to I_2^m. Let $J := 1 + j$. Let Φ^J be the result of twisting Φ by J, and $\Phi^J = 1 + \phi_J$. Then $\phi_J \in I_3^{m+1}$, as desired.

5. Proof of Propositions 3.2,3.3

5.1. **Proof of Proposition 3.2.** Let us first show that there exist quasi-Hopf algebras $H_\pm(\infty)$ defined in the same way as $H_\pm(8)$ but without the relation $x^4 = 0$. To show this, it is sufficient to check that $\Phi(\Delta \otimes \mathrm{id})\Delta(x) = (\mathrm{id} \otimes \Delta)\Delta(x)\Phi$ (the rest of the relations are straightforward). This relation is checked by a simple direct computation.

Now we must show that the ideal generated by x^4 in the quasi-Hopf algebras $H_\pm(\infty)$ is a quasi-Hopf ideal. To show this, we compute:

$$\Delta(x^2) = x^2 \otimes g + [(1+i)(p_+ \otimes p_+ + p_- \otimes p_+) + (1-i)(p_+ \otimes p_- - p_- \otimes p_-)](x \otimes x) + g \otimes x^2,$$

and hence

$$\Delta(x^4) = x^4 \otimes 1 + 1 \otimes x^4.$$

So x^4 generates a quasi-Hopf ideal, and the quotients $H_\pm(8) := H_\pm(\infty)/(x^4)$ (obviously of dimension 8) are quasi-Hopf algebras. We are done.

5.2. **Proof of Proposition 3.3.** Let $H_{+-}(\infty)$ be the amalgamated product $H_+(8) *_{H(2)} H_-(8)$ (i.e. the algebra defined as $H(32)$ but without the relation $xy + iyx = 0$). This is obviously a quasi-Hopf algebra (of infinite dimension). We must show that the principal ideal in $H_{+-}(\infty)$ generated by $z := xy + iyx$ is a quasi-Hopf ideal. This follows from the easily established relation $\Delta(z) = z \otimes 1 + g \otimes z$.

Thus, $H(32) := H_{+-}(\infty)/(z)$ is a quasi-Hopf algebra. It is easy to show that the elements $g^j x^k y^l$, $j = 0, 1$, $k, l = 0, 1, 2, 3$, form a basis in $H(32)$. Thus, $H(32)$ is 32-dimensional.

6. Proof of Theorem 3.4

Let p_+, p_- be the primitive idempotents of H_0, and let $g := p_+ - p_-$. Let Φ_0 be the associator of $\mathrm{gr}(H)$. By using a twist, we may assume that $\Phi_0 = 1 - 2p_- \otimes p_- \otimes p_-$ (the only, up to equivalence, nontrivial associator for $\mathbb{C}[\mathbb{Z}_2]$).

Let x be an element of H_1. By Theorem 2.17 in [EO], $\mathrm{Ext}^1(\varepsilon, \varepsilon) = \mathrm{Ext}^1(\chi, \chi) = 0$, thus by subsection 2.5,

$$gxg^{-1} = -x.$$

Let χ be the nontrivial character of H_0, and for $z \in H$ define

$$\xi(z) := (\chi \otimes \mathrm{id})(\Delta(z)) \text{ and } \eta(z) := (\mathrm{id} \otimes \chi)(\Delta(z)).$$

Then $\xi(g) = \eta(g) = -g$.

For $x \in H_1$ we have

$$\Delta(x) = x \otimes p_+ + p_+ \otimes x + \eta(x) \otimes p_- + p_- \otimes \xi(x).$$

By the quasi-coassociativity axiom, we have $\Phi_0(\Delta \otimes \mathrm{id})\Delta(x) = (\mathrm{id} \otimes \Delta)\Delta(x)\Phi_0$.

Lemma 6.1. *The equation $\Phi_0(\Delta \otimes id)\Delta(x) = (id \otimes \Delta)\Delta(x)\Phi_0$ is equivalent to the relations*

$$\xi^2(x) = -x,\ \eta^2(x) = -x,\ \text{and}\ \xi\eta(x) = -\eta\xi(x).$$

Proof. For degree reasons the coassociativity equation is equivalent to the system of three equations obtained by applying $\chi \otimes \chi \otimes \mathrm{id}$, $\chi \otimes \mathrm{id} \otimes \chi$, and $\mathrm{id} \otimes \chi \otimes \chi$. Since

$$(\chi \otimes \chi \otimes \mathrm{id})(\Phi_0) = (\chi \otimes \mathrm{id} \otimes \chi)(\Phi_0) = (\mathrm{id} \otimes \chi \otimes \chi)(\Phi_0) = g,$$

application of $\chi \otimes \mathrm{id} \otimes \chi$ gives

$$\begin{aligned}
\xi\eta(x) &= \xi((\mathrm{id} \otimes \chi)(\Delta(x))) \\
&= (\chi \otimes \mathrm{id} \otimes \chi)((\mathrm{id} \otimes \Delta)\Delta(x)) \\
&= (\chi \otimes \mathrm{id} \otimes \chi)(\Phi_0(\Delta \otimes \mathrm{id})\Delta(x)\Phi_0^{-1}) \\
&= g(\chi \otimes \mathrm{id} \otimes \chi)(\Delta \otimes \mathrm{id})\Delta(x))g^{-1} \\
&= g\eta\xi(x)g^{-1},
\end{aligned}$$

which is equivalent to $\xi\eta(x) = -\eta\xi(x)$.

Also, application of $\chi \otimes \chi \otimes \mathrm{id}$ yields

$$\begin{aligned}
\xi^2(x) &= \xi((\chi \otimes \mathrm{id})(\Delta(x))) \\
&= (\chi \otimes \chi \otimes \mathrm{id})((\mathrm{id} \otimes \Delta)\Delta(x)) \\
&= (\chi \otimes \chi \otimes \mathrm{id})(\Phi_0(\Delta \otimes \mathrm{id})\Delta(x)\Phi_0^{-1}) = gxg^{-1} = -x,
\end{aligned}$$

and similarly application of $\mathrm{id} \otimes \chi \otimes \chi$ yields $\eta^2(x) = -x$. $\square$

Let L_g be the operator of left multiplication by g in H_1. Then $L_g^2 = \mathrm{id}$, and $L_g\xi = -\xi L_g$, $L_g\eta = -\eta L_g$. Thus, using Lemma 6.1, we see that the operators ξ, η, L_g define an action on H_1 of the Clifford algebra Cl_3 of a $3-$dimensional inner product space.

The following lemma is standard.

Lemma 6.2. *The algebra Cl_3 is semisimple and has two irreducible representations $W_\pm$, which are both 2-dimensional. They are spanned by elements x and gx with $\eta(x) = ix$, $\xi(x) = \pm gx$.*

(Here we abuse the notation by writing g instead of L_g.)

Lemma 6.2 implies that as a Cl_3-module, $H_1 = nW_+ \oplus mW_-$.

Let $x \in W_\pm \subseteq H_1$ be an eigenvector of η with eigenvalue i. Then we have

$$\begin{aligned}
\Delta(x) &= x \otimes (p_+ \pm ip_-) + p_+ \otimes x + p_- \otimes gx \\
&= x \otimes (p_+ \pm ip_-) + 1 \otimes p_+x + g \otimes p_-x.
\end{aligned}$$

Together with Lemma 2.1 (ii), this implies the following proposition:

Proposition 6.3. *The quasi-Hopf algebra* $gr(H)$ *is generated by elements* g *of degree* 0*, and* $x_1, ..., x_n, y_1, ..., y_m$ *of degree* 1*, which satisfy the relations* $gx_j = -x_jg, gy_j = -y_jg$*,* $g^2 = 1$ *(and possibly other relations). The coproduct of* $gr(H)$ *is defined by the formulas*

$$\Delta(g) = g \otimes g,$$
$$\Delta(x_j) = x_j \otimes (p_+ + ip_-) + 1 \otimes p_+x_j + g \otimes p_-x_j,$$
$$\Delta(y_j) = y_j \otimes (p_+ - ip_-) + 1 \otimes p_+y_j + g \otimes p_-y_j.$$

Lemma 6.4. *In* $gr(H)$*, we have* $x_j^4 = y_l^4 = 0$ *for all* j, l*.*

Proof. It suffices to show that $x_j^4 = 0$, the case of y_l is obtained by changing i to $-i$. Let $x = x_j$. Using Proposition 6.3 we find that

$$\Delta(x^2) = x^2 \otimes g + [(1+i)(p_+ \otimes p_+ + p_- \otimes p_+) + (1-i)(p_+ \otimes p_- - p_- \otimes p_-)](x \otimes x) + g \otimes x^2,$$

hence

$$\Delta(x^4) = x^4 \otimes 1 + 1 \otimes x^4.$$

But a finite dimensional quasi-Hopf algebra cannot contain nonzero primitive elements (this is proved as in the Hopf case). Thus, $x^4 = 0$. □

Lemma 6.5. *The numbers* m *and* n *are either* 0 *or* 1*.*

Proof. Clearly, it suffices to show that $n = 0$ or 1. Assume the contrary, i.e. that $n \geq 2$.

Introduce the element $z := g(x_2x_1 - ix_1x_2)$. Using Proposition 6.3 it is checked directly that

$$\Delta(z) = z \otimes 1 + 1 \otimes z + 2(gx_2 \otimes p_+x_1 + ix_2 \otimes p_-x_1).$$

Let N be the smallest integer such that $z^N = 0$. It exists since gr(H) is finite dimensional. Taking the coproduct of this equation and looking at the terms of bidegree $(2N - 1, 1)$, we find

$$\sum_{k=1}^{N} z^{k-1} x_2 z^{N-k} = 0.$$

Let us now apply the coproduct to this equality, and look at the terms of bidegree $(2N-2, 1)$ which have a factor x_2 in the second component. This yields $Nz^{N-1} = 0$, which is a contradiction with the minimality of N. □

Lemma 6.6. *(i) If* $m = n = 0$*,* $gr(H) = H(2)$*.*
(ii) If $n = 1, m = 0$ *then* $gr(H) = H_+(8)$*.*
(iii) If $n = 0, m = 1$ *then* $gr(H) = H_-(8)$*.*
(iv) If $n = 1, m = 0$ *then* $gr(H) = H(32)$*.*

Proof. Statements (i)-(iii) of the lemma follow from the arguments above, since it is easy to check that the algebras $H_\pm(8)$ do not have nontrivial graded quasi-Hopf ideals which do not intersect with H_1. It remains to prove statement (iv).

Assume $m = n = 1$, and set $x := x_1, y := y_1$. Consider the element $z := xy + iyx$. A direct computation shows that

$$\Delta(z) = z \otimes 1 + g \otimes z.$$

Since $gz = zg$ and gr(H) is finite dimensional, we must have $z = 0$. Thus, gr(H) is a quotient of $H(32)$. But it is easy to check directly that $H(32)$ does not have

nontrivial graded quasi-Hopf ideals which do not intersect with H_1. We are done. $\square$

Finally, let us show that $H_+(8)$ is not twist equivalent to $H_-(8)$. Suppose they are. Then there exists a twist J for $H_-(8)$, and an isomorphism of algebras $f: H_+(8) \to H_-(8)^J$. Such an isomorphism obviously preserves filtration by powers of the radical, so we can take its degree zero part. Thus we can assume, without loss of generality, that f preserves the grading and J is of degree zero. Then f is the identity on the degree zero part, and $\Phi^J = \Phi$. So J is a twist of $\mathbb{C}[\mathbb{Z}_2]$, hence a coboundary, and thus we may assume that $J = 1 \otimes 1$ (by changing f). But then f cannot exist, since $S^2(a) = ia$ on $H_+(8)_1$ and $S^2(a) = -ia$ on $H_-(8)_1$.

This completes the proof of Theorem 3.4.

7. Relation to pointed Hopf algebras

Let $H = \oplus_{k\geq 0} H_k$ be a graded quasi-Hopf algebra with radical of codimension 2 and nontrivial associator on 1-dimensional representations.

The main result of this section is that it is actually possible to embed H into a twice bigger quasi-Hopf algebra $\tilde{H}$, which is twist equivalent to a basic Hopf algebra H' with $H'/\mathrm{Rad}(H') = \mathbb{C}[\mathbb{Z}_4]$. This fact should have generalizations to the case of general graded basic quasi-Hopf algebras, which may facilitate applications to the quasi-Hopf case of known deep results about pointed Hopf algebras.

To construct $\tilde{H}$, let us adjoin a new element a to H which is grouplike, $a^2 = g$, and $aza^{-1} = i^k z$ for $z \in H_k$. Then the algebra $\tilde{H}$ generated by a and H (of twice bigger dimension than that of H) is a quasi-Hopf algebra graded by nonnegative integers, with $\tilde{H}_0 = \mathbb{C}[\mathbb{Z}_4]$.

Proposition 7.1. *$\tilde{H}$ is twist equivalent to a Hopf algebra.*

Proof. We claim that the image of the associator Φ_0 in $\tilde{H}_0^3$ is homologically trivial. This is because the natural map $f': H^3(\mathbb{Z}_2, \mathbb{C}^*) \to H^3(\mathbb{Z}_4, \mathbb{C}^*)$ induced by the projection $f: \mathbb{Z}_4 \to \mathbb{Z}_2$ is zero. Indeed, for a cyclic group G one has $H^3(G, \mathbb{C}^*) = G^* \otimes G^*$ (functorially in G). Thus for any morphism of cyclic groups $f: G_1 \to G_2$, the induced map of third cohomology groups is $f' = f^* \otimes f^*$. In our case $f: \mathbb{Z}_4 \to \mathbb{Z}_2$, so $f^*: \mathbb{Z}_2 \to \mathbb{Z}_4$ is given by $f^*(1) = 2$. Hence, $(f^* \otimes f^*)(1 \otimes 1) = 2 \otimes 2 = 1 \otimes 4 = 0$, as desired.

Thus, by a twist in $\tilde{H}_0^2$, we can kill Φ_0. So $\tilde{H}$ is twist equivalent to a graded pointed Hopf algebra A generated in degree 1 with $A/Rad(A) = \mathbb{Z}_4$. $\square$

Example 7.2. Let $A_\pm(16)$ (for each choice of sign) be the Hopf algebra of dimension 16 generated by a, x with relations $a^4 = 1, axa^{-1} = ix, x^4 = 0$, such that a is grouplike and $\Delta(x) = 1 \otimes x + x \otimes a^{\mp 1}$.

Let $A(64)$ be the Hopf algebra of dimension 64 generated by a, x, y with relations $a^4 = 1, axa^{-1} = ix, aya^{-1} = iy, x^4 = 0, y^4 = 0, xy + iyx = 0$, such that a is grouplike and $\Delta(x) = 1 \otimes x + x \otimes a^{-1}$, $\Delta(y) = 1 \otimes y + y \otimes a$.

In other words, $A_\pm(16) = U_q(\mathfrak{b}_+)$, where $\mathfrak{b}_+$ is the Borel subalgebra in $\mathfrak{sl}_2$, and $q = \exp(\pm \pi i/4)$, while $A(64) = \mathrm{gr}(U_q(\mathfrak{sl}_2)^*)$, for the same q.

Proposition 7.3. *(i) $\widetilde{H_\pm(8)}$ is twist equivalent to $A_\pm(16)$.*

(ii) $\widetilde{H(32)}$ is twist equivalent to $A(64)$.

Proof. (i) Consider the Hopf algebra A obtained by twisting away the associator in $\widetilde{H_+(8)}$. Then $A = \bigoplus_{k\geq 0} A_k$ is a 16−dimensional graded basic (and pointed) Hopf algebra generated in degree 1. Grouplike elements of this Hopf algebra form a group $\mathbb{Z}_4$, and A_1 is a free rank 1 module over A_0 under left multiplication. So $A = A_+(16)$ or $A = A_-(16)$. To decide the sign, assume that J is a pseudotwist for $A_s(16)$, and $f : \widetilde{H_+(8)} \to A_s(16)^J$ is an isomorphism (where s is a choice of sign). Since f preserves the filtration by powers of the radical, we may assume (by taking the degree 0 part) that f preserves the grading and J has degree 0. Then $\Phi = (1 \otimes 1 \otimes 1)^J$. This implies that if $Q := \sum d_j S(e_j)$, where $J^{-1} = \sum d_j \otimes e_j$, then $QS(Q)^{-1} = a^2$. Hence the squares of the antipodes in $\widetilde{H_+(8)}$ and $A_s(16)$ differ by conjugation by a^2, i.e. by a sign on the degree 1 component. For $\widetilde{H_+(8)}$, the eigenvalue of S^2 in degree 1 is i. Thus $s = +$.

(ii) Consider the Hopf algebra A obtained by twisting away the associator in $\widetilde{H(32)}$. This is a 64-dimensional graded basic (and pointed) Hopf algebra, which is a quotient of the amalgamated product of $A_+(16)$ (generated by a, x) and $A_-(16)$ (generated by a, y). It is easy to see that $xy + iyx$ is a primitive element, so it must be zero. Thus, $A = A(64)$, which is the quotient of this amalgamated product by the relation $xy + iyx = 0$. □

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